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# Randomized and quantum algorithms yield a speed-up for initial-value problems<sup>☆</sup>

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## Abstract

Quantum algorithms and complexity have recently been studied not only for discrete, but also for some numerical problems. Most attention has been paid so far to the integration and approximation problems, for which a speed-up is shown in many important cases by quantum computers with respect to deterministic and randomized algorithms on a classical computer. In this paper, we deal with the randomized and quantum complexity of initial-value problems. For this nonlinear problem, we show that both randomized and quantum algorithms yield a speed-up over deterministic algorithms. Upper bounds on the complexity in the randomized and quantum settings are shown by constructing algorithms with a suitable cost, where the construction is based on integral information. Lower bounds result from the respective bounds for the integration problem.

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*Keywords:* Randomized algorithms; Quantum algorithms; Initial-value problems

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## 1. Introduction

Potential advantages of quantum computing over deterministic or classical randomized algorithms have been extensively studied by many authors for discrete problems, starting from Shor's paper on factorization of integers [13] and Grover's

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algorithm for searching databases [2]. Recently, a progress has also been achieved in quantum solution of numerical problems. The first paper dealing with the quantum complexity of a continuous problem was the work of Novak [12], who established matching upper and lower bounds on the quantum complexity of integration of functions from Hölder classes, based on the results on complexity of summation of real numbers from [1,10]. A general model of quantum computing for continuous problems has been developed by Heinrich [3], where the computation of a sum of real numbers is studied under various assumptions, and the results are applied to the integration problem. Another integration problem, computing path integrals, has been discussed in [14]. Recently, the approximation problem in discrete and continuous versions has also been treated in [4,5]. The linear problems of integration and approximation seem to be the only specific (and important) numerical problems discussed in the quantum setting so far.

In the randomized setting, complexity results for problems such as integration, approximation or optimization are classical, see, e.g., [11] for an overview.

In this paper, we deal with the randomized and quantum solution of initial-value problems. The complexity of this nonlinear problem was studied until now in the deterministic worst case and asymptotic settings, see, e.g., [7,8] for matching upper and lower bounds, or [9] for a discussion of complexity of initial-value problems on parallel computers.

We show in this paper that a speed-up is achieved for initial-value problems by randomized and quantum algorithms over the deterministic ones. The results are summarized in Theorem 1, where we establish upper and lower bounds on the complexity of initial-value problems in the randomized and quantum settings. The upper bounds are obtained by defining algorithms based on the deterministic integral algorithm developed in [7]. A procedure is shown allowing for an application of any algorithm for computing integrals (in the deterministic, randomized or quantum setting) to yield a new algorithm for initial-value problems in the respective setting. In the complexity analysis, the results on integration from [11,12] in the randomized and quantum settings are exploited. The comparison of the upper bounds on the randomized and quantum complexity to the worst-case complexity of initial-value problems shows that a speed-up is achieved in both nondeterministic settings. Lower bounds on the complexity refer to those for integration; the existing gap between the upper and lower bounds is discussed.

## 2. Problem formulation and results

We consider the solution of a system of ordinary differential equations with initial conditions

$$z'(t) = f(z(t)), \quad t \in [a, b], \quad z(a) = \eta, \quad (1)$$

where  $f : \mathbf{R}^d \rightarrow \mathbf{R}^d$ ,  $z : [a, b] \rightarrow \mathbf{R}^d$  and  $\eta \in \mathbf{R}^d$  ( $f(\eta) \neq 0$ ).

Nonautonomous systems  $z'(t) = f(t, z(t))$  with  $f : \mathbf{R}^{d+1} \rightarrow \mathbf{R}^d$  can be expressed in the form (1) in a usual way by adding one scalar equation

$$\begin{bmatrix} u'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 \\ f(u(t), z(t)) \end{bmatrix}$$

with an additional initial condition  $u(a) = a$ , and they are also covered by the arguments of this paper.

Given an integer  $r \geq 0$ ,  $\rho \in (0, 1]$ , and positive numbers  $D_0, D_1, \dots, D_r$  and  $H$ , we assume that the right-hand side function  $f = [f^1, \dots, f^d]^T$  belongs to the Hölder class

$$F^{r,\rho} = \{f : \mathbf{R}^d \rightarrow \mathbf{R}^d \mid f \in C^r(\mathbf{R}^d), |\partial^i f^j(y)| \leq D_i, \ i = 0, 1, \dots, r, \\ |\partial^r f^j(y) - \partial^r f^j(z)| \leq H \|y - z\|^\rho, \ y, z \in \mathbf{R}^d, \ j = 1, 2, \dots, d\}, \quad (2)$$

where  $\partial^i f^j$  represents all partial derivatives of order  $i$  of the  $j$ th component of  $f$ , and  $\|\cdot\|$  denotes the maximum norm in  $\mathbf{R}^d$ . To assure that  $f$  is a Lipschitz function, we assume that  $\rho = 1$  for  $r = 0$ .

We wish to compute a bounded function  $l$  on  $[a, b]$  that approximates the solution  $z$ . Letting  $\{x_i\}$  be the uniform partition of  $[a, b]$ ,  $x_i = a + ih$  with  $h = (b - a)/n$ , the function  $l$  will be produced by an algorithm  $\phi$ , based on approximations  $a_i(f)$  to  $z(x_i)$ ,  $i = 0, 1, \dots, n$ .

We now discuss the error and complexity models in the worst case deterministic, randomized and quantum settings. In the worst-case deterministic setting, the error of  $\phi$  at  $f$  for the problem (1) is defined by

$$e(\phi, f) = \sup_{t \in [a, b]} \|z(t) - l(t)\|$$

and the error in the class  $F^{r,\rho}$  by

$$e^{\text{worst}}(\phi, F^{r,\rho}) = \sup_{f \in F^{r,\rho}} e(\phi, f). \quad (3)$$

We assume that the values of  $f$  or its partial derivatives can be computed at given points by a subroutine. The cost of an algorithm  $\phi$  is measured by a number of subroutine calls. For a given  $\varepsilon > 0$ , by the  $\varepsilon$ -complexity of the problem,  $\text{comp}^{\text{worst}}(F^{r,\rho}, \varepsilon)$ , we mean the minimal number of subroutine calls (taken among all possible algorithms) sufficient to solve the problem with error at most  $\varepsilon$ , i.e., the minimal cost of an algorithm  $\phi$  taken among all  $\phi$  such that  $e^{\text{worst}}(\phi, F^{r,\rho}) \leq \varepsilon$ .

In the randomized setting, we allow a random selection of points at which the function  $f$  is evaluated, so that the output of an algorithm is a random variable (on a probability space  $(\Omega, \Sigma, \mathbf{P})$ ). Let the mappings  $\omega \in \Omega \rightarrow a_i^\omega(f)$  be random variables for each  $f \in F^{r,\rho}$ . By an algorithm in the randomized setting, we mean a tuple

$$\phi = (\{a_0^\omega(\cdot), a_1^\omega(\cdot), \dots, a_n^\omega(\cdot)\}_{\omega \in \Omega}, \psi), \quad (4)$$

where  $\psi$  is a mapping that produces a bounded function

$$l^\omega(t) = \psi(a_0^\omega(f), a_1^\omega(f), \dots, a_n^\omega(f))(t), \quad (5)$$

$t \in [a, b]$ , based on  $a_0^\omega(f), a_1^\omega(f), \dots, a_n^\omega(f)$ . The error of  $\phi$  at  $f$  for the problem (1) is defined by

$$e^\omega(\phi, f) = \sup_{t \in [a, b]} \|z(t) - I^\omega(t)\|. \quad (6)$$

We assume that the mapping  $\omega \in \Omega \rightarrow e^\omega(\phi, f)$  is a random variable with values in  $\mathbf{R}$ , for each  $f \in F^{r, \rho}$ . The error of  $\phi$  in the class  $F^{r, \rho}$  is given by

$$e^{\text{rand}}(\phi, F^{r, \rho}) = \sup_{f \in F^{r, \rho}} (\mathbf{E} e^\omega(\phi, f)^2)^{1/2}, \quad (7)$$

where  $\mathbf{E}$  is the expectation.

As in the worst-case setting, we measure the cost of an algorithm  $\phi$  by a number of subroutine calls that are needed to compute an approximation. For a given  $\varepsilon > 0$ , by the  $\varepsilon$ -complexity of the problem,  $\text{comp}^{\text{rand}}(F^{r, \rho}, \varepsilon)$ , we mean the minimal cost of an algorithm  $\phi$  taken among all  $\phi$  such that  $e^{\text{rand}}(\phi, F^{r, \rho}) \leq \varepsilon$ .

In the quantum setting, the output of an algorithm is a random variable (taking a finite number of values), but the reason of randomness is different than that in the randomized setting. On a quantum computer, where, roughly speaking, basic objects are qubits (elements of a two-dimensional complex space  $H_1$ ) and allowed operations are unitary transformations of the tensor product of a number of copies of  $H_1$ , randomness is a result of quantum measurement operations. For a detailed description of the framework of numerical quantum computing one is referred to [3], where the notions of quantum measurement, quantum algorithm and complexity are defined and thoroughly discussed, and applications to summation and integration problems are studied. For a condensed discussion of randomized and quantum settings, in particular for the integration problem, one is also referred to [6].

By a quantum algorithm  $\phi$  for solving our problem we mean a tuple (4), where  $a_i^\omega(f)$  are random approximations, in the quantum sense, to  $z(x_i)$  for each  $f$ . The error of  $\phi$  at  $f$  is defined by (6).

Let  $0 < \delta < \frac{1}{2}$ . The error of  $\phi$  in  $F^{r, \rho}$  in the quantum setting is defined [3] by

$$e^{\text{quant}}(\phi, F^{r, \rho}, \delta) = \sup_{f \in F^{r, \rho}} \inf \{ \alpha \mid \mathbf{P}\{e^\omega(\phi, f) > \alpha\} \leq \delta \}. \quad (8)$$

Note that for a given  $\varepsilon > 0$  the bound  $e^\omega(\phi, f) \leq \varepsilon$  holds with probability at least  $1 - \delta$  for each  $f$  iff  $e^{\text{quant}}(\phi, F^{r, \rho}, \delta) \leq \varepsilon$ .

In the quantum setting the value of  $\delta$  is usually set to  $\delta = \frac{1}{4}$ . The error probability can then be reduced to any  $\delta$  by computing (componentwise) the median of  $c \log 1/\delta$  repetitions of the algorithm, where  $c$  is a positive number independent of  $\delta$ , see [4], Lemma 3. For our problem, the procedure of increasing the probability of success can be applied at different levels which influences a logarithmic part of the cost of an algorithm, so that we shall describe it in more detail and discuss after the proof of Theorem 1.

On a quantum computer, the right-hand side function  $f$  can be accessed through a query that returns, for a given point, a value of a component of  $f$ . Roughly speaking, a query on a class of real functions is defined as a transformation  $Q$  that associates with each function  $p$  a unitary mapping  $Q_p$  defined on a Hilbert quantum space. For

a detailed discussion of what is meant by “returning a value” of a function, and how a query is implemented in the quantum setting, the reader is referred to [3] or [12]. The cost of an algorithm  $\phi$  is measured by a number of quantum queries that are needed to compute an approximation along with a number of classical evaluations of  $f$  or its partial derivatives. For a given  $\varepsilon > 0$ , by the quantum  $\varepsilon$ -complexity of the problem,  $\text{comp}^{\text{quant}}(F^{r,\rho}, \varepsilon, \delta)$ , we mean the minimal cost of a quantum algorithm  $\phi$  taken among all  $\phi$  such that  $e^{\text{quant}}(\phi, F^{r,\rho}, \delta) \leq \varepsilon$ .

We prove in this paper upper and lower bounds on the randomized and quantum complexity of initial-value problems (1). Upper bounds, summarized in the following theorem, will be next compared to the known lower bounds on deterministic complexity to show that a speed-up is achieved in both settings. Lower bounds in the randomized and quantum settings are derived by a simple argument in the case  $d \geq 2$ , and are also included. For  $d \geq 2$  (and also in the case of nonautonomous equations with  $d = 1$ ), finding lower bounds on the complexity of (1) can be reduced to the similar problem for integration. For autonomous problems with  $d = 1$  a straightforward reduction to the integration problem is not possible, and the question about lower bounds is then still open (a similar difficulty is successfully faced in the worst-case setting in [7]).

We take below  $\log = \log_2$  (although the base of the logarithm is not crucial).

**Theorem 1.** *For the problem (1), we have that*

$$\text{comp}^{\text{rand}}(F^{r,\rho}, \varepsilon) = O\left(\left(\frac{1}{\varepsilon}\right)^{\frac{r+\rho+3/2}{(r+\rho+1/2)(r+\rho+1)}} \log \frac{1}{\varepsilon}\right), \quad (9)$$

$$\text{comp}^{\text{quant}}(F^{r,\rho}, \varepsilon, \delta) = O\left(\left(\frac{1}{\varepsilon}\right)^{\frac{r+\rho+2}{(r+\rho+1)^2}} \left(\log \frac{1}{\varepsilon} + \log \frac{1}{\delta}\right)\right). \quad (10)$$

For  $d \geq 2$ ,

$$\text{comp}^{\text{rand}}(F^{r,\rho}, \varepsilon) = \Omega\left(\left(\frac{1}{\varepsilon}\right)^{\frac{1}{r+\rho+1/2}}\right), \quad (11)$$

and, for  $0 < \delta \leq \frac{1}{4}$ ,

$$\text{comp}^{\text{quant}}(F^{r,\rho}, \varepsilon, \delta) \geq \text{comp}^{\text{quant}}(F^{r,\rho}, \varepsilon, \frac{1}{4}) = \Omega\left(\left(\frac{1}{\varepsilon}\right)^{\frac{1}{r+\rho+1}}\right). \quad (12)$$

The constants in the “ $O$ ” and “ $\Omega$ ” notation only depend on the class  $F^{r,\rho}$ , and are independent of  $\varepsilon$  and  $\delta$ .

Upper bounds (9) and (10) will be derived by defining suitable algorithms, while the lower bounds (11) and (12) are equal to those on the complexity of randomized or quantum computation of integrals of a function of one variable.

Before giving the proof, we make some comments on these results. If the values of  $f$  or its partial derivatives can only be accessed, the deterministic worst-case complexity of the problem (1) is of the order  $\varepsilon^{-1/(r+\rho)}$ , see Theorem 3 in the next section. Since

$$\frac{r+\rho+2}{(r+\rho+1)^2} < \frac{r+\rho+3/2}{(r+\rho+1/2)(r+\rho+1)} < \frac{1}{r+\rho}, \quad (13)$$

both randomized and quantum computation yield an improvement over the deterministic setting over the entire range of  $r$  and  $\rho$  (we neglect the logarithmic factors). For instance, if  $r=0$  and  $\rho=1$ , the worst-case complexity in the deterministic setting is of the order  $\varepsilon^{-1}$ , in the randomized setting it is bounded from above by  $\varepsilon^{-5/6}$ , while on a quantum computer by  $\varepsilon^{-3/4}$ .

The lower bounds coincide with those for the integration problem, see (16) and (17). To see what the size of the gap between the bounds is, note that the reciprocal of the exponent in  $1/\varepsilon$  in the quantum case is such that

$$\lim_{r \rightarrow \infty} \left( \frac{(r+\rho+1)^2}{r+\rho+2} - (r+\rho) \right) = 0,$$

so that it behaves for large  $r$  as  $r+\rho$ , while the lower bound depends on  $r+\rho+1$ . In the randomized setting, the reciprocal of the exponent in  $1/\varepsilon$  behaves for large  $r$  like  $r+\rho$ , while the lower bound depends on  $r+\rho+1/2$ .

We now recall results on randomized and quantum computation of integrals, as well as those on deterministic solution of initial-value problems.

### 3. Randomized and quantum computation of integrals and deterministic solution of initial-value problems

Quantum complexity of integration has been first studied by Novak [12]. The problem is to approximate the integral

$$I(g) = \int_{[0,1]^s} g(x) dx, \quad (14)$$

for functions  $g : [0, 1]^s \rightarrow \mathbf{R}$  from a Hölder class

$$\begin{aligned} \tilde{F}^{r,\rho} &= \{g \in C^r([0, 1]^s) \mid |g(y)| \leq \tilde{D}_0, |\partial^r g(y) - \partial^r g(z)| \\ &\leq \tilde{H} \|y - z\|^\rho, \quad y, z \in [0, 1]^s\} \end{aligned} \quad (15)$$

with  $r \geq 0$ ,  $0 < \rho \leq 1$ , where  $\partial^r g$  represents all partial derivatives of order  $r$  of  $g$ .

The error of an algorithm  $\phi$  at  $g$  for the integration problem (14) in the randomized and quantum settings is defined by

$$e^\omega(\phi, g) = |I(g) - A^\omega(g)|,$$

where  $A^\omega(g)$  is the output of  $\phi$  (in the worst-case setting the definition is the same, only the output is deterministic). The other definitions of errors in the class of

functions and complexity remain the same as for the problem (1), with  $F^{r,\rho}$  replaced by  $\tilde{F}^{r,\rho}$ .

Based on the results on the computation of the mean of  $n$  numbers given by Brassard et al. [1] (upper bound) and Nayak and Wu [10] (lower bound), Novak [12] showed the following result in the quantum setting. For the result in the randomized setting, see [11, p. 62]. Let  $\gamma = (r + \rho)/s$ .

**Theorem 2** (Novak [11,12]). *For the problem (14) we have*

$$\text{comp}^{\text{rand}}(\tilde{F}^{r,\rho}, \varepsilon) \asymp \varepsilon^{-1/(\gamma+1/2)}, \quad (16)$$

$$\text{comp}^{\text{quant}}(\tilde{F}^{r,\rho}, \varepsilon, \tfrac{1}{4}) \asymp \varepsilon^{-1/(\gamma+1)}. \quad (17)$$

An upper bound in (16) can be achieved by random algorithms with a finite number of output values.

Consider now the solution of initial-value problem (1) in the deterministic setting. This problem has been considered in a number of papers, see, e.g., [7] or [8]. The following result is a straightforward modification of Corollary 4.1 from [7]. The modification is needed, since the class of functions considered in [7], consisting of  $r$  times continuously differentiable functions with bounded derivatives, is to be replaced with the Hölder class  $F^{r,\rho}$ . For the modification in lower bounds, one is referred to the proof of Theorem 3.1 from [8], where the functions  $g_k$  in the construction must be replaced by suitable functions from the class  $F^{r,\rho}$ . The upper bounds will be derived again in the sequel, as a by-product in the proof of Theorem 1.

**Theorem 3** (Kacewicz [7]). *In the deterministic setting, the complexity of (1) satisfies:*

- *if the values of  $f$  or its partial derivatives are only accessible, then*

$$\text{comp}^{\text{worst}}(F^{r,\rho}, \varepsilon) \asymp \varepsilon^{-1/(r+\rho)}, \quad (18)$$

- *if arbitrary linear functionals are accessible, then*

$$\text{comp}^{\text{worst}}(F^{r,\rho}, \varepsilon) \asymp \varepsilon^{-1/(r+\rho+1)}. \quad (19)$$

(The lower bound in (19) holds true not only for linear functionals, but also for a class of nonlinear functionals, see [7].) Relation (19) will play an important role in the proof of Theorem 1, while (18) will serve as a point of reference in evaluating a speed-up obtained due to randomization or due to quantum computations.

Let us now recall the algorithm that leads to the upper bound in (19). It requires the computation of integrals of  $f$ , and is defined as follows.

Take  $y_0^* = \eta$ . Given  $y_i^*$  ( $y_i^* \cong z(x_i)$ ), we let  $\bar{z}_i^*(t)$  be the solution of the problem

$$\bar{z}'(t) = f(\bar{z}(t)), \quad t \in [x_i, x_{i+1}], \quad \bar{z}(x_i) = y_i^* \quad (20)$$

and we set

$$l_i^*(t) = \sum_{j=0}^{r+1} \frac{1}{j!} \bar{z}_i^{*(j)}(x_i) (t - x_i)^j, \quad t \in [x_i, x_{i+1}]. \quad (21)$$

Then, we define

$$y_{i+1}^* = y_i^* + \int_{x_i}^{x_{i+1}} f(l_i^*(t)) dt, \quad (22)$$

$i = 0, 1, \dots, n-1$ , and finally

$$l(t) = l_i^*(t) \quad \text{for } t \in [x_i, x_{i+1}]. \quad (23)$$

(The function  $l$  is piecewise continuous. It is also possible to define it to be continuous on  $[a, b]$ .)

#### 4. Randomized and quantum solution of initial-value problems

We now define randomized and quantum algorithms for the solution of (1). Let  $w_i^*$  be a polynomial

$$w_i^*(y) = \sum_{j=0}^r \frac{1}{j!} f^{(j)}(y_i^*) (y - y_i^*)^j, \quad (24)$$

where  $f^{(j)}(y_i^*)z^j$  is meant to be the value of the  $j$ -linear operator  $f^{(j)}(y_i^*)$  at  $(z, z, \dots, z)$  ( $j$  times). The values of  $w_i^*$  can be computed through evaluation of partial derivatives of components of  $f$  of order  $0, 1, \dots, r$ . Equality (22) can be equivalently written as

$$y_{i+1}^* = y_i^* + \int_{x_i}^{x_{i+1}} w_i^*(l_i^*(t)) dt + h^{r+\rho+1} \int_0^1 g_i(u) du, \quad (25)$$

where

$$g_i(u) = \frac{1}{h^{r+\rho}} (f(l_i^*(x_i + uh)) - w_i^*(l_i^*(x_i + uh))), \quad (26)$$

for  $u \in [0, 1]$  and  $i = 0, 1, \dots, n-1$ . Properties of  $g_i$  will play an important role in the proof of Theorem 1. We state them in the following:

**Lemma.** *The functions  $g_i$  belong to  $C^{(r)}([0, 1])$ , the derivatives of  $g_i$  of order  $0, 1, \dots, r$  are bounded by constants that depend only on the parameters of the class  $F^{r,\rho}$  (and are independent of  $i$ ,  $y_i^*$  and  $h$ ), and*

$$\|g_i^{(r)}(u) - g_i^{(r)}(\bar{u})\| \leq \tilde{H} |u - \bar{u}|^\rho,$$

$u, \bar{u} \in [0, 1]$ , for some constant  $\tilde{H}$  depending only on the parameters as above.

**Proof.** For a fixed  $i$ , we write  $g_i(u)$  as  $g_i(u) = \frac{1}{h^{r+\rho}} R(l_i^*(x_i + uh))$ , where  $R(y) = f(y) - w_i^*(y)$ . The derivatives of  $g_i$  of order  $k = 1, 2, \dots, r$  can be expressed in terms



of the derivatives of  $R$  (with respect to  $y$ ) and  $l_i^*$  (with respect to  $t$ ) as follows:

$$g_i^{(k)}(u) = \frac{1}{h^{r+\rho-k}} (R^{(k)}(l_i^*(x_i + uh))(l_i^{*'}(x_i + uh))^k + S(u)), \quad (27)$$

where  $S(u)$  is a sum of terms of the form  $R^{(j)}(l_i^*(x_i + uh))\alpha_1 \dots \alpha_j$ ,  $j = 1, 2, \dots, k-1$ , the arguments  $\alpha_l$  being derivatives of order  $1, 2, \dots, k$  of  $l_i^*$  with respect to  $t$ , evaluated at  $t = x_i + uh$ . Hence,  $g_i \in C^{(r)}([0, 1])$ . To show boundedness of  $g_i$  and its derivatives, observe that by Taylor's remainder formula we have for  $k = 0, 1, \dots, r-1$  that

$$R^{(k)}(y) = \int_0^1 (f^{(r)}(\theta y + (1-\theta)y_i^*) - f^{(r)}(y_i^*)) (y - y_i^*)^{r-k} \frac{(1-\theta)^{r-k-1}}{(r-k-1)!} d\theta,$$

which yields that

$$\|R^{(k)}(l_i^*(x_i + uh))\| \leq Ch^{r+\rho-k}$$

for  $u \in [0, 1]$ , with a constant  $C$  depending only on the parameters of the class  $F^{r,\rho}$  (the symbol  $C$  will be used in the sequel to denote possibly different constants). The same bound obviously holds for  $k = r$ . This, together with (27), implies the boundedness of  $g_i^{(k)}$  by constants that only depend on the parameters of the class  $F^{r,\rho}$ .

It remains to show that  $g_i^{(r)}$  is a Hölder function in  $[0, 1]$ . For  $r = 0$  the desired inequality immediately follows from the Hölder property of  $f$ . For  $r \geq 1$  we use (27) with  $k = r$ . We first note that  $S(u)$  is a continuously differentiable function with first derivative bounded by  $Ch^\rho$ , where  $C$  only depends on the parameters of the class  $F^{r,\rho}$ . Hence,  $S(u)/h^\rho$  is a Lipschitz function with the constant  $C$ .

The first term in (27) has the form  $(1/h^\rho)A(u)a(u)^r$ , where  $A(u) = R^{(r)}(l_i^*(x_i + uh))$  and  $a(u) = l_i^{*'}(x_i + uh)$ . To show that this is a Hölder function, we use the inequality

$$\begin{aligned} & (1/h^\rho) \|A(u)a(u)^r - A(\bar{u})a(\bar{u})^r\| \\ & \leq (1/h^\rho) (\|A(u) - A(\bar{u})\| \|a(u)\|^r + \|A(\bar{u})\| \|a(u) - a(\bar{u})\| \\ & \quad \times \sum_{l=0}^{r-1} \|a(u)\|^l \|a(\bar{u})\|^{r-1-l}). \end{aligned}$$

To complete the proof it suffices to use bounds on  $\|A(u)\|$ ,  $\|a(u)\|$  and  $\|a(u) - a(\bar{u})\|$  that result directly from the properties of  $f$ , and the inequality

$$\|A(u) - A(\bar{u})\| \leq Ch^\rho |u - \bar{u}|^\rho,$$

$u, \bar{u} \in [0, 1]$ , with a constant  $C$  depending only on the parameters of the class  $F^{r,\rho}$ .  $\square$

The algorithm (deterministic, randomized or quantum) for solving (1) is defined as follows (we omit the argument  $\omega$  in random variables). Let  $a_0(f) = y_0 = \eta$ . Given  $a_i(f) = y_i$ , we consider functions  $g_i$  defined by (26) for  $y_i$  (that is, the polynomials  $l_i^*$  and  $w_i^*$  based on  $y_i^*$  are replaced by the polynomials  $l_i$  and  $w_i$  based on  $y_i$ ), and compute some approximations  $A_i(f)$  to the integrals  $\int_0^1 g_i(u) du$  in (25). The

algorithm is defined by setting  $a_{i+1}(f) = y_{i+1}$ , where

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} w_i(l_i(t)) dt + h^{r+\rho+1} A_i(f) \quad (28)$$

and the approximation on  $[x_i, x_{i+1}]$  is given by

$$l(t) = l_i(t), \quad (29)$$

$i = 0, 1, \dots, n-1$ . Finally, we set  $\psi(a_0(f), \dots, a_n(f))(t) = l(t)$  for  $t \in [a, b]$ .

The approximations  $A_i(f)$  may be obtained by deterministic, randomized or quantum algorithms. In the randomized and quantum settings, we demand random variables  $A_i(f)$  to satisfy

$$\left\| \int_0^1 g_i(u) du - A_i(f) \right\| \leq \varepsilon_1 \quad \text{with probability at least } (1 - \delta)^{1/n} \quad (30)$$

for  $i = 0, 1, \dots, n-1$  (and all  $y_i$ ), for some  $\varepsilon_1$ . It will be shown later on that a satisfactory choice is  $\varepsilon_1 = h$ .

For illustration, we specify the algorithm above in the case  $r = 0$ . It may be considered as a modification of Euler's method and is defined as follows. Given  $y_i$ , we compute

$$y_{i+1} = y_i + hf(y_i) + h^{1+\rho} A_i(f),$$

where  $A_i(f)$  is a (deterministic, randomized or quantum) approximation to  $(1/h^\rho) \int_0^1 (f(y_i + uhf(y_i)) - f(y_i)) du$  with error at most  $\varepsilon_1$ , and probability at least  $(1 - \delta)^{1/n}$  (in nondeterministic cases),  $i = 0, 1, \dots, n-1$ . The approximation to  $z = z(t)$  is defined on  $[x_i, x_{i+1}]$  by  $l(t) = y_i + f(y_i)(t - x_i)$ .

**Proof of Theorem 1.** Consider first the quantum setting. We use the quantum algorithm of Novak to compute  $A_i(f)$  (componentwise in a statistically independent way). Due to (17) with  $s = 1$ , we have that for each  $i$  the inequality

$$\left\| \int_0^1 g_i(u) du - A_i(f) \right\| \leq \varepsilon_1 \quad (31)$$

holds with probability at least  $\frac{3}{4}$ , and the quantum query cost  $O(\varepsilon_1^{-1/(r+\rho+1)})$ . (Since  $g_i$  has  $d$  components, only the constant in the “ $O$ ” notation is different than that in (17), which is a result of computing the median of a suitable number of repetitions to increase the probability of success in each component to  $(\frac{3}{4})^{1/d}$ .) Taking componentwise the median of  $k$  repetitions,

$$k = \Theta \left( \log \frac{1}{1 - (1 - \delta)^{1/n}} \right),$$

we arrive at an approximation  $A_i(f)$  (the same symbol is used to denote this new approximation) such that (30) is satisfied. Since

$$\log \frac{1}{1 - (1 - \delta)^{1/n}} \leq c(\log n + \log 1/\delta)$$

(where  $c$  is independent of  $n$  and  $\delta$ ), the cost of computing  $A_i(f)$  is of order  $O(\varepsilon_1^{-1/(r+\rho+1)} (\log n + \log 1/\delta))$ . Thus, with the cost  $O(\varepsilon_1^{-1/(r+\rho+1)} n(\log n + \log 1/\delta))$  we assure that the bounds

$$\left\| \int_0^1 g_i(u) du - A_i(f) \right\| \leq \varepsilon_1 \quad \text{for } i = 0, 1, \dots, n-1 \quad (32)$$

hold simultaneously with probability at least  $1 - \delta$ . The total cost of the algorithm additionally includes  $O(n)$  classical subroutine calls necessary in the deterministic part of the algorithm.

Let  $e_i = z(x_i) - y_i$ . Since the solution of (1) satisfies

$$z(x_{i+1}) = z(x_i) + \int_{x_i}^{x_{i+1}} f(z(t)) dt, \quad (33)$$

we get from (28) that

$$\begin{aligned} e_{i+1} = e_i &+ \int_{x_i}^{x_{i+1}} (f(z(t)) - f(l_i(t))) dt \\ &+ \int_{x_i}^{x_{i+1}} (f(l_i(t)) - w_i(l_i(t))) dt - h^{r+\rho+1} A_i(f). \end{aligned} \quad (34)$$

We shall derive from (34) a difference inequality for  $\|e_i\|$ . Let  $\bar{z}_i$  denote the solution of the local problem (20) with the initial condition  $\bar{z}_i(x_i) = y_i$ . By triangle inequality,

$$\|f(z(t)) - f(l_i(t))\| \leq \|f(z(t)) - f(\bar{z}_i(t))\| + \|f(\bar{z}_i(t)) - f(l_i(t))\|. \quad (35)$$

To estimate the first term, we note that the dependence of the solution on initial condition yields that

$$\|z(t) - \bar{z}_i(t)\| \leq \exp(Lh) \|z(x_i) - y_i\|, \quad t \in [x_i, x_{i+1}], \quad (36)$$

where  $L$  is the Lipschitz constant of  $f$ . By Taylor's formula we get

$$\bar{z}_i(t) - l_i(t) = \int_0^1 (\bar{z}_i^{(r+1)}(\theta t + (1 - \theta)x_i) - \bar{z}_i^{(r+1)}(x_i))(t - x_i)^{r+1}(1 - \theta)^r / r! d\theta,$$

$t \in [x_i, x_{i+1}]$ . Evaluating  $\bar{z}_i^{(r+1)}$  in terms of partial derivatives of  $f$  and using much the same type of arguments as those used in the proof of Lemma, we check that the Hölder condition for  $\partial^r f^j$  for any  $j$  implies that  $\bar{z}_i^{(r+1)}$  is a Hölder function with exponent  $\rho$  and a constant that only depends on the parameters of the class  $F^{r,\rho}$  (and is independent of  $i, y_i, n$ ). Hence,

$$\|\bar{z}_i(t) - l_i(t)\| \leq Mh^{r+\rho+1}, \quad (37)$$

$t \in [x_i, x_{i+1}]$ , where the constant  $M$  only depends on the parameters of the class  $F^{r,\rho}$ .

The last two inequalities together with (34), (35) and (32) yield the relation (satisfied with probability at least  $1 - \delta$ )

$$\|e_{i+1}\| \leq \|e_i\|(1 + hL \exp(Lh)) + LMh^{r+\rho+2} + h^{r+\rho+1}\varepsilon_1, \quad (38)$$

$i = 0, 1, \dots, n-1$ . Take now  $\varepsilon_1 = h$ . By solving difference inequality (38), we get that

$$\max_{0 \leq i \leq n} \|e_i\| = O(h^{r+\rho+1}) \quad (39)$$

with probability at least  $1 - \delta$ . Finally, for any  $t \in [x_i, x_{i+1}]$  we have due to (36), (37) and (39) that

$$\|z(t) - l(t)\| = \|z(t) - l_i(t)\| \leq \|z(t) - \bar{z}_i(t)\| + \|\bar{z}_i(t) - l_i(t)\| = O(h^{r+\rho+1}), \quad (40)$$

with the constant depending only on the parameters of the class  $F^{r,\rho}$ , and probability at least  $1 - \delta$ .

The quantum query cost of the considered algorithm is of order  $O(n^{(r+\rho+2)/(r+\rho+1)}(\log n + \log 1/\delta))$ . For  $\varepsilon > 0$ , we now take the minimal  $n$  for which the upper bound in (40) does not exceed  $\varepsilon$ ,  $n \asymp (1/\varepsilon)^{1/(r+\rho+1)}$ , and the desired upper bound (10) on the complexity follows.

In the randomized setting we use (16). Take the algorithm  $\phi$  for approximating integrals with  $s = 1$  for which the upper bound on the complexity is achieved, and let  $e^{\text{rand}}(\phi, \tilde{F}^{r,\rho}) \leq \varepsilon_1/2$ . By the Chebyshev inequality we have that

$$\mathbf{P}\left\{\left|\int_0^1 g(x) dx - A^\omega(g)\right| > \varepsilon_1\right\} \leq 1/4,$$

for any  $g \in \tilde{F}^{r,\rho}$ , where  $A^\omega(g)$  is the output of the algorithm. As in the quantum case, the inequality (31) holds with probability at least  $\frac{3}{4}$ , but the cost is now  $O(\varepsilon_1^{-1/(r+\rho+1/2)})$ . Proceeding further on similarly as in the quantum case, one gets that the bound (40) holds with probability at least  $1 - \delta$ , and cost  $O(n^{(r+\rho+3/2)/(r+\rho+1/2)}(\log n + \log 1/\delta))$ . We denote the resulting random algorithm for initial value problems by  $\tilde{\phi}$ .

Take now  $n$  to be the minimal number for which the upper bound in (40) is at most  $\varepsilon/2$ ,  $n \asymp (1/\varepsilon)^{1/(r+\rho+1)}$ , and observe that

$$\begin{aligned} \mathbf{E}e^\omega(\tilde{\phi}, f)^2 &= \int_{e^\omega(\tilde{\phi}, f) > \varepsilon/2} e^\omega(\tilde{\phi}, f)^2 d\mathbf{P}(\omega) + \int_{e^\omega(\tilde{\phi}, f) \leq \varepsilon/2} e^\omega(\tilde{\phi}, f)^2 d\mathbf{P}(\omega) \\ &\leq K^2\delta + \varepsilon^2/4 \end{aligned}$$

for all  $f \in F^{r,\rho}$ , where  $K$  is a positive constant that depends only on the parameters of the class  $F^{r,\rho}$  such that  $e^\omega(\tilde{\phi}, f) \leq K$  (such a constant exists, since we may assume that  $|A^\omega(g)| \leq 2\tilde{D}_0$ ; otherwise we would get a better approximation by setting  $A^\omega(g) = 0$ ). Hence, the algorithm  $\tilde{\phi}$  with  $\delta = 3\varepsilon^2/4K^2$  satisfies  $e^{\text{rand}}(\tilde{\phi}, F^{r,\rho}) \leq \varepsilon$ . Looking at the cost of  $\tilde{\phi}$  we see that the upper bound (9) is proven.

Consider now the deterministic setting. The known upper bounds on the complexity can be derived again as follows. If the values of  $f$  or its partial derivatives can only be accessed, then we are able to approximate  $\int_0^1 g_i(u) du$  within the error  $\varepsilon_1$  with cost  $O(\varepsilon_1^{-1/(r+\rho)})$ . Since  $n$  integrals are to be approximated, the total cost of the algorithm (for  $\varepsilon_1 = h$ ), with the error bound (39), is of order  $O(n \cdot n^{1/(r+\rho)})$ . This proves the upper bound in (18). If exact computation of the integrals is allowed in the model, then we simply take  $\varepsilon_1 = 0$ , which leads to the upper bound in (19).

We now pass to lower bounds. Consider the quantum setting. Let  $g \in \tilde{F}^{r,\rho}$  with  $r \geq 1$  and  $\rho \in (0, 1]$ , or  $r = 0$  and  $\rho = 1$ , and with  $s = 1$ . Consider a two-dimensional problem

$$\begin{cases} u'(t) = 1, \\ v'(t) = g(u(t)), \quad t \in [0, 1], \end{cases} \quad (41)$$

with initial conditions  $u(0) = 0, v(0) = 0$ . The function  $g$  can be extended to  $\mathbf{R}$  such that the right-hand side function in (41) belongs to the class  $F^{r,\rho}$  with suitably chosen parameters. The solution of (41) is given by  $u(t) = t$  and  $v(t) = \int_0^t g(s) ds$ .

Let  $\varepsilon > 0$  and let  $\phi$  be any quantum algorithm for solving (1) with error  $e^{\text{quant}}(\phi, F^{r,\rho}, \frac{1}{4}) \leq \varepsilon$ , and quantum query cost  $c(\varepsilon)$ . When applied to (41), the algorithm  $\phi$  gives an approximation to  $v(1) = \int_0^1 g(t) dt$  with error at most  $\varepsilon$ , with probability at least  $\frac{3}{4}$ . Due to the lower bound for integration in (17) the cost must be at least of order  $\varepsilon^{-1/(r+\rho+1)}$  queries on  $g$ , which yields that  $c(\varepsilon) = \Omega(\varepsilon^{-1/(r+\rho+1)})$ , and proves lower bound (12).

In the randomized setting we use similar arguments to show (11), adjusted to the error formula (7). Since the same arguments apply in both settings for any  $d \geq 2$ , the proof of Theorem 1 is completed.  $\square$

Let us finally note that the logarithmic factor in the upper bound (10) depends on what stage the median is computed at in the algorithm. If we ask the inequality (30) to hold with probability at least  $(\frac{3}{4})^{1/n}$ , proceed up to the final step (40) with  $\delta = \frac{1}{4}$ , and after that compute the median of  $c \log 1/\delta$  repetitions of the entire algorithm, then the factor  $\log 1/\varepsilon + \log 1/\delta$  in (10) would be replaced by  $\log 1/\varepsilon \cdot \log 1/\delta$ .

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